

SUBHARMONICITY OF THE MODULUS OF QUASIREGULAR HARMONIC MAPPINGS

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ABSTRACT. In this note we determine all numbers $q \in \mathbf{R}$ such that $|u|^q$ is a subharmonic function, provided that u is a K -quasiregular harmonic mappings in an open subset Ω of the Euclidean space \mathbf{R}^n .

1. INTRODUCTION

By $|\cdot|$ we denote the Euclidean norm in \mathbf{R}^n and let Ω be a region in \mathbf{R}^n . In this paper we consider K -quasiregular harmonic mappings, where $K \geq 1$. We recall that a harmonic mapping $u(x) = (u_1(x), \dots, u_n(x)) : \Omega \rightarrow \mathbf{R}^n$ with formal differential matrix

$$Du(x) = \{\partial_i u_j(x)\}_{i,j=1}^n$$

is K -quasiregular if

$$(1.1) \quad K^{-1}|Du(x)|^n \leq J_u(x) \leq Kl(Du(x))^n, \quad \text{for all } x \in \Omega,$$

where J_u is the Jacobian of u at x ,

$$|Du| := \max\{|Du(x)h| : |h| = 1\},$$

and

$$l(Du) := \min\{|Du(x)h| : |h| = 1\}.$$

See [7, p. 128] for the definition of quasiregular mappings in more general setting. A quasiregular homeomorphism is called quasiconformal.

Let $0 < \lambda_1^2 \leq \lambda_2^2 \leq \dots \leq \lambda_n^2$ be the eigenvalues of the matrix $Du(x)Du(x)^t$. Here $Du(x)^t$ is the transpose of the matrix $Du(x)$. Then

$$(1.2) \quad J_u(x) = \prod_{k=1}^n \lambda_k,$$

$$(1.3) \quad |Du| = \lambda_n$$

and

$$(1.4) \quad l(Du) = \lambda_1.$$

For the Hilbert-Schmidt norm of the matrix $Du(x)$, defined by

$$\|Du(x)\| = \sqrt{\text{Trace}(Du(x)Du(x)^t)}$$

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we have

$$(1.5) \quad \|Du(x)\| = \sqrt{\sum_{k=1}^n \frac{\partial u}{\partial x_k} \bullet \frac{\partial u}{\partial x_k}} = \sqrt{\sum_{k=1}^n \left| \frac{\partial u}{\partial x_k} \right|^2}$$

and

$$(1.6) \quad \|Du(x)\| = \sqrt{\sum_{k=1}^n \lambda_k^2}.$$

Here \bullet denotes the inner product between vectors. From (1.1), for a quasiregular mapping we have

$$(1.7) \quad \frac{\lambda_n}{\lambda_k}, \frac{\lambda_k}{\lambda_1} \leq K, \quad k = 1, \dots, n.$$

It is well known that if $u = (u_1, \dots, u_n)$ is a harmonic mapping defined in a region Ω of the Euclidean space \mathbf{R}^n , then $|u|^p$ is subharmonic for $p \geq 1$, and that, in the general case, is not subharmonic for $p < 1$. Let us prove this well-known fact. If u is harmonic, then by a result in [4, Lemma 1.4.] (see also [3, Eq. (4.9)-(4.11)])

$$\Delta|u| = |u| \left\| D \left(\frac{u}{|u|} \right) \right\|^2.$$

So $\Delta|u| \geq 0$ for those points x , such that $u(x) \neq 0$. If $u(a) = 0$, then we consider the harmonic mapping $u_m(x) = u(x) + (1/m, 0, \dots, 0)$. Then $u_m(a) \neq 0$, and $\Delta|u_m(x)| \geq 0$ in some neighborhood of a . It follows from the definition of subharmonic functions that the uniform limit of a convergent sequence of subharmonic functions is still subharmonic. Since $|u_m(x)| \rightarrow |u(x)|$, it follows that $|u|$ is subharmonic in a . Since the function $g(s) = s^p$, is convex for $p \geq 1$, we obtain that $|u|^p$ is subharmonic providing that u is harmonic. (For the above facts we refer to [2, Chapter 2]).

Recently, several authors have proved the following two propositions, which is the motivation for our study.

Proposition 1.1. [5] *If f is a K -quasiregular harmonic map in a plane domain, then $|f|^q$ is subharmonic for $q \geq 1 - K^{-2}$.*

Proposition 1.2. [1] *If f is a K -quasiregular harmonic map in a space domain, then $|f|^q$ is subharmonic for some $q = q(K, n) \in (0, 1)$.*

This paper is continuation of [1] in which Proposition 1.1 was extended to the n -dimensional setting. In [1] the authors prove only the existence of an exponent $q \in (0, 1)$ without giving the minimal value of q . Here we improve Proposition 1.2 by giving the optimal value of q . Our proof is completely different from those given in [1] and [5]. Moreover for the first time we consider the case $q < 0$.

Our proof is based on the following well-known explicit computation.

Proposition 1.3. [6, Ch. VII 3, p.217]. *Let $u = (u_1, \dots, u_n) : \Omega \rightarrow \mathbf{R}^n$, be harmonic, let $\Omega_0 = \Omega \setminus u^{-1}(0)$, let $q \in \mathbf{R}$. Then for $x \in \Omega_0$*

$$\Delta |u|^q = q \left[|u|^{q-2} \sum_{k=1}^n |\nabla u_k|^2 + (q-2)|u|^{q-4} \sum_{k=1}^n \left(u \bullet \frac{\partial u}{\partial x_k} \right) \right].$$

Proof. Write $v := |u|^q = (u_1^2 + \dots + u_n^2)^p$, for $p := q/2$. A direct computation gives

$$\begin{aligned} v_{x_1} &= p(u_1^2 + \dots + u_n^2)^{p-1} \cdot (2u_1 u_{1x_1} + \dots + 2u_n u_{nx_1}) \\ &= q(u_1^2 + \dots + u_n^2)^{p-1} \cdot (u_1 u_{1x_1} + \dots + u_n u_{nx_1}), \end{aligned}$$

and further

$$\begin{aligned} v_{x_1 x_1} &= q \{ 2(p-1)(u_1^2 + \dots + u_n^2)^{p-2} \cdot (u_1 u_{1x_1} + \dots + u_n u_{nx_1})^2 + \\ &\quad + (u_1^2 + \dots + u_n^2)^{p-1} \cdot [u_1 u_{1x_1 x_1} + (u_{1x_1})^2 + \dots + u_n u_{nx_1 x_1} + (u_{nx_1})^2] \}. \end{aligned}$$

Therefore

$$\begin{aligned} \Delta v &= v_{x_1 x_1} + \dots + v_{x_n x_n} \\ &= q \{ |u|^{q-2} [(u_1 \Delta u_1 + \dots + u_n \Delta u_n) + (\sum_{k=1}^n u_{1x_k}^2 + \dots \\ &\quad + \sum_{k=1}^n u_{nx_k}^2)] + (q-2)|u|^{q-4} \sum_{k=1}^n \left(\sum_{j=1}^n u_j u_{jx_k} \right)^2 \} \\ &= q \{ |u|^{q-2} (\sum_{k=1}^n u_{1x_k}^2 + \dots + \sum_{k=1}^n u_{nx_k}^2) + (q-2)|u|^{q-4} \sum_{k=1}^n \left(\sum_{j=1}^n u_j \cdot \frac{\partial u_j}{\partial x_k} \right)^2 \} \\ &= q |u|^{q-4} \{ |u|^2 \sum_{j=1}^n \left(\sum_{k=1}^n u_{jx_k}^2 \right) + (q-2) \sum_{k=1}^n \left(\sum_{j=1}^n u_j \cdot \frac{\partial u_j}{\partial x_k} \right)^2 \} \\ &= q |u|^{q-4} \{ |u|^2 \sum_{j=1}^n |\nabla u_j|^2 + (q-2) \sum_{k=1}^n \left(u \bullet \frac{\partial u}{\partial x_k} \right)^2 \}. \end{aligned}$$

□

2. MAIN RESULT

Theorem 2.1. *Let u be K -quasiregular harmonic in $\Omega \subset \mathbf{R}^n$. Then the mapping $g(x) = |u(x)|^q$ is subharmonic in*

- (1) Ω for $q \geq \max\{1 - \frac{n-1}{K^2}, 0\}$;
- (2) $\Omega \setminus u^{-1}(0)$, for $q \leq 1 - (n-1)K^2$.

Moreover for $1 - (n-1)K^2 < q < 1 - \frac{n-1}{K^2}$, there exists a K -quasiconformal harmonic mapping such that $|u|^q$ is not subharmonic.

Remark 2.2. If $n = 2$ then $1 - \frac{n-1}{K^2} = 1 - K^{-2}$. Thus Theorem 2.1 is an extension of Proposition 1.1.

Remark 2.3. In the case $1 \leq K \leq \sqrt{n-1}$ the function $|u|^q$ is subharmonic for all $q > 0$.

Proof of Theorem 2.1. Let us fix such a map $u : \Omega \rightarrow \mathbf{R}^n$ and set $\Omega_0 = \Omega \setminus u^{-1}\{0\}$. We have to find all positive real numbers q such that $\Delta|u|^q \geq 0$ on Ω_0 . Since u is quasiregular, the set $Z = \{x \in \Omega_0 : \det Du(x) = 0\}$ has measure zero (see [7]), it is also closed since u is smooth. In particular, $\Omega_1 = \Omega_0 \setminus Z$ is dense in Ω_0 and thus it suffices to prove that $\Delta|u|^q \geq 0$ on Ω_1 . From Proposition 1.3, we obtain

$$(2.1) \quad \Delta|u|^q = q \left[|u|^{q-2} \|Du\|^2 + (q-2)|u|^{q-4} \left| \sum_{j=1}^n u_j \nabla u_j \right|^2 \right].$$

So we find all real q such that

$$q \left(|u|^{q-2} \|Du\|^2 + (q-2)|u|^{q-4} \left| \sum_{j=1}^n u_j \nabla u_j \right|^2 \right) \geq 0.$$

If $q \geq 2$, then $\Delta|u|^q \geq 0$. Assume that $q \geq 0$ and $q < 2$ such that

$$\left| \sum_{j=1}^n u_j(x) \nabla u_j(x) \right|^2 \leq \frac{1}{2-q} |u(x)|^2 \|Du(x)\|^2, \quad x \in \Omega_1.$$

After normalization, we see that it suffices to find all constants $q < 2$ such that

$$(2.2) \quad \sup_{z \in S^{n-1}} \left| \sum_{j=1}^n z_j \nabla u_j(x) \right|^2 \leq \frac{1}{2-q} \|Du(x)\|^2, \quad x \in \Omega_1.$$

Let $0 < \lambda_1^2 \leq \lambda_2^2 \leq \dots \leq \lambda_n^2$ be the eigenvalues of the matrix $Du(x)Du(x)^t$. Then

$$(2.3) \quad \sup_{z \in S^{n-1}} \left| \sum_{j=1}^n z_j \nabla u_j(x) \right|^2 = \lambda_n^2$$

$$(2.4) \quad \inf_{z \in S^{n-1}} \left| \sum_{j=1}^n z_j \nabla u_j(x) \right|^2 = \lambda_1^2$$

and

$$(2.5) \quad \|Du(x)\|^2 = \sum_{k=1}^n \lambda_k^2.$$

Because u is K -quasiregular from (1.7) we have

$$(2.6) \quad \frac{\lambda_n}{\lambda_k} \leq K, \quad k = 1, \dots, n-1.$$

Thus (2.2) can be written as

$$(2.7) \quad \lambda_n^2 \leq \frac{1}{2-q} \sum_{k=1}^n \lambda_k^2.$$

By (2.5) and (2.6) we get that, the inequality (2.7) is satisfied whenever

$$(2.8) \quad \frac{1}{1 + \frac{n-1}{K^2}} \leq \frac{1}{2-q}$$

i.e.

$$(2.9) \quad \max \left\{ 0, 1 - \frac{n-1}{K^2} \right\} \leq q < 2.$$

If $q < 0$, then we should have

$$(2.10) \quad \inf_{z \in S^{n-1}} \left| \sum_{j=1}^n z_j \nabla u_j(x) \right|^2 \geq \frac{1}{2-q} \|Du(x)\|^2, \quad x \in \Omega_1,$$

i.e.

$$2-q \geq \sum_{k=1}^n \frac{\lambda_k^2}{\lambda_1^2}.$$

Because u is K -quasiregular from (1.7)

$$(2.11) \quad \frac{\lambda_k}{\lambda_1} \leq K, \quad k = 2, \dots, n.$$

Thus if

$$(2.12) \quad q \leq 1 - (n-1)K^2,$$

then (2.10) holds. To finish the proof we need the following lemma.

Lemma 2.4. *For any $1 - (n-1)K^2 < q < 1 - \frac{n-1}{K^2}$ there is a (linear) harmonic K -quasiconformal mapping u such that $|u|^q$ is not subharmonic.*

Proof of Lemma 2.4. Assume first that $q > 0$. We will consider linear mapping $u : \mathbf{R}^n \rightarrow \mathbf{R}^n$ defined by

$$(2.13) \quad u(x_1, \dots, x_{n-1}, x_n) = (x_1, \dots, x_{n-1}, Kx_n),$$

where $K \geq 1$. It is obviously harmonic and K -quasiconformal. If we put this mapping in formula (2.1) we get

$$[(n-1) + K^2]|u|^2 + (q-2) \left| \sum_{j=1}^{n-1} x_j e_j + K^2 e_n x_n \right|^2 \geq 0$$

which is equivalent to

$$(n-1+K^2) \left[\sum_{n=1}^{j-1} x_j^2 + K^2 x_n^2 \right] + (q-2) \left| \sum_{j=1}^{n-1} x_j e_j + K^2 e_n x_n \right|^2 \geq 0.$$

By choosing $x_1 = \dots = x_{n-1} = 0$ and $x_n = 1$, we obtain

$$(n-1+K^2)K^2 \geq (2-q)K^4$$

which is equivalent to

$$q \geq 1 - \frac{n-1}{K^2}.$$

For $q < 0$ we consider the linear mapping $u : \mathbf{R}^n \rightarrow \mathbf{R}^n$ defined by

$$(2.14) \quad u(x_1, \dots, x_{n-1}, x_n) = (x_1, \dots, x_{n-1}, x_n/K).$$

□

To finish the proof we only need to take $\tilde{u} = u|_{\Omega}$, where u is defined in (2.13) respectively in (2.14). □

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REFERENCES

- [1] ARSENOVIĆ, M., BOŽIN, V. AND MANOJLOVIĆ, V.: *Moduli of continuity of harmonic quasiregular mappings in B^n* : Potential Analysis DOI: 10.1007/s11118-010-9195-8.
- [2] HAYMAN, W. K., KENNEDY, P. B.: *Subharmonic functions*, I Academic Press, London-New York, 1976, xvii+284 pp.
- [3] KALAJ, D.: *A priori estimate of gradient of a solution to certain differential inequality and quasiconformal mappings*. arXiv:0712.3580v3. 24 pp.
- [4] KALAJ, D.: *On the univalent solution of PDE $\Delta u = f$ between spherical annuli*. J. Math. Anal. Appl. **327** (2007), no. 1, 1–11.
- [5] KOJIĆ, V., PAVLOVIĆ, M.: *Subharmonicity of $|f|^p$ for quasiregular harmonic functions with applications*. J. Math. Anal. Appl. **342** (2008), 742–746 .
- [6] STEIN, E.M.: *Singular integrals and differentiability properties of functions*. Princeton Mathematical Series, No. 30 Princeton University Press, Princeton, N.J. 1970 xiv+290 pp.
- [7] VUORINEN, M.: *Conformal geometry and quasiregular mappings*. Lecture Notes in Math., 1319. Springer, Berlin (1988).

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